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THE METHOD OF DISCRETE SINGULARITIES IN PLANE PROBLEMS OF THE THEORY OF ELASTICITY WITH NON-SMOOTH BOUNDARIES*

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The numerical solution of a class of elasticity-theory problems that is broader compared with that considered earlier in /1-3/ is investigated, namely, plane problems with non-smooth boundaries by the method of discrete singularities (MDS). The MDS is the direction of numerical solution of boundary value problems that is substantially the Tikhonov regularization method /4/ based on boundary singular integral equations (SIE). It is best to combine the MDS with the method of finite elements in computations of geometrically complex objects when the solution for the low level superelements is obtained by using the MDS.

The MDS includes the reduction of the problem to SIE, the parametric assignment of the contour, the investigation of the SIE properties, smoothing of the SIE kernels, extraction of the unique SIE solution, justification of the selection of two matched systems of points on the contour, passage from SIE to a system of linear algebraic equations and assurance of its determinancy and non-degeneracy, analysis of the convergence of the solution, and the application of quadrature formulas for Cauchy-type integrals.

An elastic isotropic homogeneous medium is considered that occupies a simply-connected domain D with a piecewise-smooth closed contour Γ on a plane. The solution of plane elasticity theory problems in a known way /5, 6/ is reduced to determining two analytic functions, φ, ψ, say.

The non-smoothness of the boundary influences the realization of the MDS, however, the essence of the method is conserved. The purpose of the paper being published is to study and give a foundation for what is new in the MDS for plane elasticity theory problems with non-smooth boundaries as compared with the smooth boundary case in question.

1. To reduce two fundamental problems (in displacements and stresses) to SIE to the first kind, it was proposed /2/ to represent the analytic functions φ and ψ in terms of one complex function ω in the following form (L is a smooth contour)

$$\varphi(z)=\frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau-z}\,d\tau,\quad \psi(z)=\frac{1}{2\pi i}\int_L \frac{\varkappa_k\overline{\omega(\tau)}-\bar{\tau}\omega'(\tau)}{\tau-z}\,d\tau,\quad z\in D \tag{1.1}$$

The derivative was calculated from (1.1) by means of the formula

$$\varphi'(z)=\frac{1}{2\pi i}\int_L \frac{\omega'(\tau)}{\tau-z}\,d\tau \tag{1.2}$$

obtained from (1.1) after differentiation with respect to z and integration by parts, for the derivation of the SIE from the boundary condition /5, 6/. However, in certain problems with non-smooth boundaries the function $\omega(\tau)$ may turn out to be such that the integral (1.2) of $\omega'(\tau)$ does not exist. Then the derivative $\varphi'(z)$ must be evaluated as the integral of the function $\omega(\tau)$

$$\varphi'(z)=\frac{1}{2\pi i}\int_\Gamma \frac{\omega(\tau)}{(\tau-z)^2}\,d\tau \tag{1.3}$$

We formulate the following problem: it is required to obtain the SIE for the function ω from the boundary condition irrespective of whether the derivative $\varphi'(z)$ is calculated by (1.2) or (1.3).

From the boundary condition of the problem and (1.1) and (1.2) we arrive at a SIE of the first kind /2/ on the contour Γ if the relationship (1.2) is true for it

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$$\frac{\varkappa_k}{\pi i}\int_\Gamma \omega(\tau)\,\mathrm{Re}\left(\frac{d\tau}{\tau-t}\right)+\frac{1}{2\pi i}\int_\Gamma \overline{\omega(\tau)}\,d\left(\frac{\tau-t}{\bar\tau-\bar t}\right)=f_k(t),\quad t\in\Gamma \tag{1.4}$$

and we also arrive at the SIE (1.4) with $c=\varkappa_k$ from (1.3) and relationships /7/ analogous to (1.1)

$$\varphi(z)=\frac{1}{2\pi i}\int_\Gamma \frac{\omega(\tau)}{\tau-z}\,d\tau \tag{1.5}$$

$$\psi(z)=\frac{1}{2\pi i}\left[\bar c\int_\Gamma \frac{\overline{\omega(\tau)}}{\tau-z}\,d\tau+\int_\Gamma \frac{\omega(\tau)}{\tau-z}\,d\bar\tau-\int_\Gamma \frac{\bar\tau\omega(\tau)}{(\tau-z)^2}\,d\tau\right],\quad z\in D$$

Indeed, we determine the limit values $\varphi^{\pm}(t)$ of the function $\varphi(z)$ at the arbitrary point $t\in\Gamma$ /8/

$$\varphi^{\pm}(t)=\alpha^{\pm}\omega(t)+\frac{1}{2\pi i}\int_\Gamma \frac{\omega(\tau)}{\tau-t}\,d\tau;\quad \alpha^{+}=1-\frac{\alpha}{2\pi},\quad \alpha^{-}=-\frac{\alpha}{2\pi} \tag{1.6}$$

where the plus and minus superscripts correspond to D^+ (the internal problem) and D^- (the external problem).

In the usual way /9/ we determine the right (left) tangential vector with origin at the arbitrary point t of the contour Γ as the limit position of the secant for the neighbourhood $(t, t+\Delta t)$ (for the neighbourhood $(t-\Delta t, t)$) of the contour Γ traversed positively and as $\Delta t\to 0$. Then α is the angle between the right and left tangential vectors defined counter-clockwise. For smooth points of the contour Γ we have $\alpha=\pi$ for angular points of the contour $\alpha\in\,]0,\pi[\,\bigcup\,]\pi,2\pi[$, and for reentry points $\alpha=0,\ 2\pi$ for the cusp directed, respectively, to the right or left for a positive traversal of Γ for which the domain D remains on the left.

As in (1.6) we find the limit value $[t\overline{\varphi'(t)}+\overline{\psi(t)}]^{\pm}$ of the functions $z\overline{\varphi'(z)}+\overline{\psi(z)}$

$$[t\overline{\varphi'(t)}+\overline{\psi(t)}]^{\pm}=c\alpha^{\pm}\omega(t)-\frac{1}{2\pi i}\left[c\int_\Gamma \frac{\omega(\tau)}{\bar\tau-\bar t}\,d\bar\tau+\int_\Gamma \frac{\overline{\omega(\tau)}}{\bar\tau-\bar t}\,d\tau-\int_\Gamma \frac{(\tau-t)\,\overline{\omega(\tau)}}{(\bar t-\bar\tau)^2}\,d\bar\tau\right] \tag{1.7}$$

and from the boundary condition for the two fundamental problems $(k=1,2)$

$$\varkappa_k\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}=f_k(t) \tag{1.8}$$

taking (1.6) and (1.7) into account, we arrive at a SIE for the complex function $\omega(t)$

$$(\varkappa_k-c)\,\alpha^{\pm}\omega(t)+\frac{1}{2\pi i}\left[\varkappa_k\int_\Gamma \frac{\omega(\tau)}{\tau-t}\,d\tau+c\int_\Gamma \frac{\omega(\tau)}{\bar\tau-\bar t}\,d\bar\tau+\int_\Gamma \overline{\omega(\tau)}\,d\left(\frac{\tau-t}{\bar\tau-\bar t}\right)\right]=f_k(t) \tag{1.9}$$

(the notation is the same as in /1-3, 7/). For $c=\varkappa_k$ the SIE (1.9) reduces to the SIE (1.4).

2. Let us examine the properties of the SIE (1.4). Its eigenfunctions depend on $\varkappa_k$ and have been studied earlier /2/. Thus, for $k=1$ a complex constant is the eigenfunction of (1.4) and for $k=2$, the function $ia\tau$ in addition, where a is a real constant.

To investigate the properties of the kernel of the SIE (1.4) we change to a new integration variable whose differential is a continuous function on Γ. We can take as such, say, the parameter η of the mutually single-valued parametric assignment of the contour Γ

$$x=x(\eta),\quad y=y(\eta),\quad \eta\in[0,2\pi] \tag{2.1}$$

where $x(\eta), y(\eta)$ and their derivatives with respect to η are 2π periodic functions. The contour Γ is closed, therefore, the kernels of the SIE (1.4) are periodic functions. Taking this into account, we extract the singularity from the singular kernel A_0 of the SIE (1.4) in the form of a sum of a Hilbert kernel and a periodic function $A(\eta,\xi)$ whose regularity is proved /2/ for a Lyapunov contour L (the derivatives with respect to η are denoted by primes)

$$\mathrm{Re}\left(\frac{d\tau}{\tau-t}\right)=A_0(\eta,\xi)\,d\eta=\frac{x'(x-x_0)+y'(y-y_0)}{r^2}\,d\eta=\frac{1}{2}\left[\mathrm{ctg}\,\frac{\eta-\xi}{2}+A(\eta,\xi)\right]d\eta \tag{2.2}$$

$$\eta,\xi\in[0,2\pi],\ \tau=x+iy,\ t=x_0+iy_0,\ x_0=x(\xi),\ y_0=y(\xi)$$

Here

$$A(\eta,\xi)=\frac{2}{r^2}[x'a(\eta,\xi)+y'b(\eta,\xi)]+ \tag{2.3}$$
$$\operatorname{ctg}\frac{\eta-\xi}{2}\left[\frac{4}{r_0^2}(x'x_0'+y'y_0')-1\right]$$
$$a(\eta,\xi)=x-x_0-x_0'\sin(\eta-\xi),\ b(\eta,\xi)=y-y_0-$$
$$y_0'\sin(\eta-\xi)$$
$$r^2=(x-x_0)^2+(y-y_0)^2,\quad r_0^2=r^2\sin^{-2}\frac{\eta-\xi}{2}$$

For $\eta=\xi$ the function $A(\eta,\xi)$ is obtained from (2.3) by passing to the limit

$$A(\eta,\eta)=(x'x''+y'y'')/S'^2=(x''\cos\beta+y''\sin\beta)/S'= \tag{2.4}$$
$$S''/S'$$

$$S'=\sqrt{x'^2+y'^2},\ S''=(x'x''+y'y'')/S' \tag{2.5}$$

where β is the slope of the tangent to Γ at the point η.
We write the SIE (1.4) taking (2.2) into account

$$\varkappa_K\int_0^{2\pi}\omega(\eta)\left[\operatorname{ctg}\frac{\eta-\xi}{2}+A(\eta,\xi)\right]d\eta+\int_0^{2\pi}\overline{\omega(\eta)}B(\eta,\xi)\,d\eta= \tag{2.6}$$
$$2\pi i f_K(\xi)$$

$$B(\eta,\xi)=A_1(\eta,\xi)+iA_2(\eta,\xi) \tag{2.7}$$
$$A_1(\eta,\xi)=4r^{-4}(x-x_0)(y-y_0)[x'(y-y_0)-y'(x-x_0)]$$
$$A_2(\eta,\xi)=-2r_0^{-4}[(x-x_0)^2-(y-y_0)^2][x'(y-y_0)-$$
$$y'(x-x_0)]$$

After passing to the limit and taking account of (2.5) we obtain from (2.7) for $\eta=\xi$

$$B(\eta,\eta)=ie^{i2\beta}(y''\cos\beta-x''\sin\beta)/S'=iKS'e^{i2\beta} \tag{2.8}$$

where K is the curvature of the contour Γ at a point with parameter η.
Writing the SIE (1.4) in the form (2.6) shows the kind of SIE being investigated (with singular kernel $\operatorname{ctg}\frac{\eta-\xi}{2}$) and separates two functions $A(\eta,\xi)$ and $B(\eta,\xi)$ on which the non-smoothness of the contour Γ has an influence.

We determine Λ_2 as the class of curves L for which x'',y'' are functions of the class H /8/ and $S'\neq 0$. If the contour $L\in\Lambda_2$, then the functions $A(\eta,\xi)$ and $B(\eta,\xi)$ in (2.6) are of class H /1/.

If

$$\Gamma=\bigcup_{l=1}^{m}L_l,\quad L_l\in\Lambda_2$$

and there are angular points of reentry points η_i on Γ (i.e., points at which the direction of the tangential vector undergoes a discontinuity of the first kind): $\beta(\eta_i+0)=\beta_i^+$; $\beta(\eta_i-0)=\beta_i^-$ and $\beta_i^+\neq\beta_i^-$), then the functions $A(\eta,\xi), B(\eta,\xi)$ can have discontinuities of the first kind because of the derivatives x', y' according to (2.3) and (2.7). For $\eta=\eta_i=\xi$ the nature of the discontinuity of the function $B(\eta,\eta)$ will be determined, in conformity with (2.8), by values of the limits to the left and right at the point η_i of the curvature

$$K=(x'y''-y'x'')/S'^3$$

of the curve Γ and the limit values of the angle β. And according to (2.4), the nature of the discontinuity of the function $A(\eta,\eta)$ is determined by the limit values of the ratio S''/S'.

We shall consider the contour Γ with those singularities whose presence under the allowed parametric assignment of the contour $(S'\neq 0)$ results in discontinuities of the functions $A(\eta,\xi)$ and $B(\eta,\xi)$ of only the first kind. Then the solution of the SIE (2.6) is the function $\omega(\eta)$ which can have not more than a logarithmic singularity /8, 10/ at these points.

This same assertion holds for those points of Γ at which the right side of the SIE (2.6) has a discontinuity of the first kind, and the functions $A(\eta,\xi), B(\eta,\xi)$ are continuous. If both the right side of the SIE (2.6) and the functions $A(\eta,\xi)$, $B(\eta,\xi)$, simultaneously undergo a discontinuity of the first kind at a point of the contour Γ then prediction of the behaviour of the solution of the SIE requires additional investigation.

The real and imaginary parts of the SIE (2.6) yield a system of SIE in the functions $\omega_R,\ \omega_I\ (\omega=\omega_R+i\omega_I)$

$$\int_0^{2\pi} \omega_R(\eta)\left\{\varkappa_k\left[\operatorname{ctg}\frac{\eta-\xi}{2}+A(\eta,\xi)\right]+A_1(\eta,\xi)\right\}d\eta+\int_0^{2\pi}\omega_I(\eta)A_2(\eta,\xi)\,d\eta=-2\pi f_{kI}(\xi) \tag{2.9}$$

$$\int_0^{2\pi} \omega_I(\eta)\left\{\varkappa_k\left[\operatorname{ctg}\frac{\eta-\xi}{2}+A(\eta,\xi)\right]-A_1(\eta,\xi)\right\}d\eta+\int_0^{2\pi}\omega_R(\eta)A_2(\eta,\xi)\,d\eta=2\pi f_{kR}(\xi)$$

We note that both smooth and unsmooth right-hand sides are possible in the system SIE (2.9). For instance, for problems in stresses $(k=2)$ /2/, the functions f_{2I}, f_{2R} generally have discontinuities of the first kind at those points of the contour L where the concentrated loads are applied. In the absence of these concentrated loads the functions f_{2I}, f_{2R} are continuous at all points of the contour Γ.

3. Extraction of the unique solution of the SIE (1.4) in the case of a smooth contour L was achieved /2, 9/ by introducing additional integral conditions whose purpose was to "reduce" the SIE by using integral conditions with a spectrum. A numerical comparison was performed /11/ between this and the traditional /6, 12/ approaches in which the solvability of the SIE was achieved because of the inclusion of additional components. The advantages of the additional conditions is shown in /11/ since the effectiveness of the additional components, associated mainly with an error in the calculations, drops abruptly as the order n of the system of linear algebraic equations corresponding to the SIE being solved numerically, incrases.

The integral conditions "reducing" the SIE with a spectrum can be written as before /1-3/ as well as in the form

$$\int_\Gamma \omega(\tau)\,dS=0,\quad k=1,2;\quad \int_\Gamma\left(\frac{\omega}{\tau}-\frac{\bar\omega}{\bar\tau}\right)dS=0,\quad k=2 \tag{3.1}$$

where dS is the length element of the contour Γ. The first equation in (3.1) "does not pass" the proper solution of the SIE (1.4), which equals a complex constant, but the second is $ia\tau$ where a is a real constant.

If $\omega(\tau)$ is a continuous function, the integrands in (3.1) have no discontinuities at angular points of the contour Γ. This is indeed the advantage of conditions (3.1) as compared with the conditions proposed earlier in /1-3/ whose integrands can have discontinuities of the first kind in the case of a non-smooth contour Γ.

The introduction of conditions (3.1) assumes that non-smoothness of the contour Γ does not influence the eigenfunctions of the SIE (1.4). This assumption is later verified numerically for the solution of specific problems.

4. The passage from the SIE to a system of algebraic equations requires additional constraints on the selection of the two matched systems of the point η_i and η_j on the contour Γ. These constraints are a generalization of the assertion formulated earlier /2/ that design points (DP) ξ_j should be placed at points of discontinuity of the first kind of the right side of (1.4) (or (2.9)), while the value of the right side is taken equal to half the sum of its limits on the left and right of this point.

Indeed, we extract the Hilbert kernel in the SIE (1.4) /2, 10/. It is known for it /13/ that if the density $\varphi(\eta)$ is expanded in a Fourier series

$$\varphi(\eta)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos k\eta+b_k\sin k\eta),\quad \varphi(\eta)\in L_2[0,2\pi] \tag{4.1}$$

then

$$I(\xi)\equiv\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\eta-\xi}{2}\varphi(\eta)\,d\eta=\sum_{k=1}^{\infty}(-a_k\sin k\xi+b_k\cos k\xi)=f(\xi) \tag{4.2}$$

The series obtained is a Fourier series for the function $f(\xi)$, and consequently, if $f(\xi)$ has a discontinuity ξ_0 of the first kind, then

$$I(\xi_0)={}^1/_2\,[f(\xi_0-0)+f(\xi_0+0)] \tag{4.3}$$

since by passing to the limit as $n\to\infty$ in the equation

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\eta-\xi_0}{2}S_n(\varphi)\,d\eta=S_n(f)$$

where $S_n(\varphi)$ is a partial sum of the series (4.1), we obtain (4.3) by virtue of the boundedness of this operator in $L_2[0, 2\pi]$. Hence, the analogous assertion follows for the SIE (1.4) with the regular part, i.e.,

$$\frac{\varkappa_k}{\pi i}\int_\Gamma \omega(\tau)\,\mathrm{Re}\left(\frac{d\tau}{\tau - t_0}\right) + \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\tau)}\, d\left(\frac{\tau - t_0}{\bar\tau - \bar t_0}\right) = \tag{4.4}$$
$$^1/_2\,[f_k(t_0 - 0) + f_k(t_0 + 0)], \quad t_0 \in \Gamma$$

where t_0 is a point of discontinuity of the first kind for the right side of the SIE (1.4).

It follows from (4.4) that the left side of the SIE takes on a strictly definite value at the point t_0. This is taken into account in the numerical solution when going over from the SIE to the discrete analogue so that t_0 is called a PT (a collocation point) of the SIE (1.4).

As the total number n of PT changes on the contour Γ the PT at the points of discontinuity of the first kind on the right side of (1.4) are conserved and will be called fixed PT, unlike the other PT.

As is usual /1, 2/, two matched systems of points η_i and ξ_j are used when going over from the SIE to the discrete analogue. Investigations executed numerically for the solutions of specific problems enabled us to establish that angular points of the contour Γ must be taken as fixed PT (see the explanation in Sect.7). Other versions of the selection of the points η_i, ξ_j result is substantial **perturbations** of the SIE solution in the neighbourhoods of the singularities, as a rule. This result was verified in problems in which it is known that the solution in stresses has no perturbations near angular points of Γ, for instance, in the case of the internal problem $k = 2$ about the multilateral uniform tension of a rectangular domain D.

Thus, on going over to the discrete analogue of the SIE, fixed PT should be fixed at the angular points of the contour Γ and points of discontinuity of the first kind of the right side of the SIE (1.4). The location of the fixed PT governs the greatest possible partition spacing (or the minimum number of points) on the contour Γ when constructing two matched systems of points η_i and ξ_j equidistant in the assignment parameter of the contour Γ. We note that the distribution of the points η_i, ξ_j on the contour Γ can turn out to be substantially non-uniform.

5. The transfer from the SIE (1.4) and the integral conditions (3.1) to a system of algebraic equations in problems on a non-smooth simply-connected contour Γ $(\Gamma = L_1 + \ldots + L_m$, $L_l \in \Lambda_2$, $l = 1, \ldots, m)$ is made after partitioning the integrals over Γ in (1.4) and (3.1) into a sum of integrals over L_l. This is preceded by partitioning of the domain of definition of the variables η, ξ belonging to the segment $[0, 2\pi]$, into m sections, in proportion to the curve lengths L_l, say, and selecting the partitioning step on each of the sections taking the presence of fixed PT on the contour Γ taken into account. Consequently, we go from (2.9) and (3.1) over to a system of linear algebraic equations (summation over i from 1 to n)

$$\begin{aligned}
&\Sigma\,[\omega_{Rn}(\eta_i)(2\varkappa_k A_0 + A_1) + \omega_{In}(\eta_i) A_2]\,\alpha_{ij} + \\
&\quad \beta_1 + (k-1)\beta_3 \cos\xi_j = -2\pi f_{kI}(\xi_j) \\
&\Sigma\,[\omega_{Rn}(\eta_i) A_2 + \omega_{In}(\eta_i)(2\varkappa_k A_0 - A_1)]\,\alpha_{ij} + \\
&\quad \beta_2 + (k-1)\beta_3 \sin\xi_j = 2\pi f_{kR}(\xi_j) \\
&\Sigma\omega_{Rn}(\eta_i) S'(\eta_i) = 0, \quad \Sigma\omega_{In}(\eta_i) S'(\eta_i) = 0 \\
&(k-1)\,\Sigma\,[\omega_{Rn}(\eta_i) B_1(\eta_i) - \omega_{In}(\eta_i) B_2(\eta_i)] = 0; \\
&k = 1, 2; \quad j = 1, \ldots, n
\end{aligned} \tag{5.1}$$

The quantities α_{ij} are determined by quadrature formulas /1/ depending on the selection of the distribution law of the two matched systems of points η_i, ξ_j on the contour Γ; for η_i, ξ_j distributions with a constant step along Γ we have

$$\alpha_{ij} = \frac{2\pi}{n}, \quad B_1(\eta_i) = \frac{y}{r^2} S'(\eta_i), \quad B_2(\eta_i) = \frac{x}{r^2} S'(\eta_i), \quad S'(\eta_i) = \sqrt{x'^2 + y'^2}.$$

The unknowns β_1, β_2, β_3, as indeed in the case of the smooth contour L /1-3/, are regularizing factors and are introduced to explain the determinacy of system (5.1) in which the number of equations $N = 2n + 3$ equals the number of unknown values of the functions ω_R, ω_I at n points each plus the three unknown constants β_1, β_2, β_3. The last three unknowns tend to zero as $n \to \infty$ /2/.

6. Convergence of the solution of system (5.1) of order N to the solution of the SIE (1.4) with conditions (3.1) follows from the proof in /1/ and the application of formulas of the rectangle type to evaluate the Cauchy-type integral

$$\frac{1}{2\pi i}\int_L \frac{\omega(\tau)}{\tau - t}\,d\tau \approx \frac{1}{2\pi i}\sum_{i=1}^{n}\frac{\omega(\tau_i)\,\Delta\tau_i}{\tau_i - t_k}, \quad t, \tau_i, \quad t_k \in L \tag{6.1}$$

over a closed contour L, where τ_1, t_k are two matched mutually alternating systems of points on the contour L obtained there by using a uniform partition over the parameter for parametric assignment of the contour L. If L is here a smooth closed contour and the function $\omega(\tau) \in H(\alpha)$ on L (Hölder type), then convergence of the integral (6.1) is uniform of the type $O(\ln n/n^\alpha)$; if $L = \Gamma$ has an angle or the function $\omega(\tau)$ has discontinuities of the first kind, then convergence of the integral (6.1) is uniform of the same type outside the neighbourhood of the angles and integral, as a whole, on the whole contour Γ /1/.

Satisfaction of the integral convergence of the solution of system (5.1) for problems $k = 2$ means, in particular, convergence in N determinable stresses at all points of the domain D and uniform convergence in Γ outside neighbourhoods of angles means convergence in N determinable streses in $D \cup \Gamma$ outside the neighbourhoods of the angles. Calculation of the stresses in the neighbourhoods of the angles requires uniform convergence of the numerical solution of the SIE in the neighbourhood of the angles of the contour Γ.

7. We call smoothing of the kernels A and B of the SIE (2.6) the elimination of discontinuities of the first kind.

A method is proposed for smoothing the kernels A and B of the SIE (2.6) that is based on elimination of discontinuities of the first kind, determined from (2.3) and (2.7) for $\eta \neq \xi$. At the same time, in conformity with (2.4) and (2.8), in the general case discontinuities remain for $\eta = \xi$. However, according to (2.4) the value of the function is $A(\eta, \eta) = 0$ in the angle formed by arcs of circles when they are parametrized in the form $x = R\cos\eta$, $y = R\sin\eta$. And the value of the function $B(\eta, \eta)$ is zero, according to (2.8), in the angle formed by straight lines since the curvature is $K = 0$ there. In the cases mentioned the functions $A(\eta, \eta), B(\eta, \eta)$ have no discontinuities of the first kind at angular points of Γ.

The proposed smoothing of the kernels A, B is achieved for $\eta \neq \xi$ by the special selection of the parametric assignment of the contour Γ and the two matched systems of points on it.

The contour Γ can be given parametrically by both a single law along its whole length and by different laws, for instance by its parametric equations $x = x(\eta)$, $y = y(\eta)$ on each curve L_l. If such a parametric assignment of the contour Γ is found, then as before, two matched, systems of points η_i, ξ_j equidistant in the assignment parameter for the contour Γ are given which are used according to the method of discrete singularities to find the numerical solution of the SIE that can possess uniform convergence everywhere in Γ as computations show.

We consider the example of the parametric assignment of the contour Γ in the form of power series in the roots of a Chebyshev polynomial of the first kind.

On the curve $L_l \in \Lambda_2$ let the parameter be $\eta \in (\eta_l, \eta_{l+1})$ while the coordinate $x = x_l$ varies between a and b. Then the coordinate $x_l(\eta)$ of the curve L_l can be given parametrically in a linear approximation in the form

$$\begin{gathered} x_l(\eta) = c_0 + c_1\cos\eta_{0l}, \quad \eta_{0l} = \alpha_0 + \alpha_1\eta \\ \eta = \eta_l,\ x_l(\eta_l) = a,\ \alpha_0 + \alpha_1\eta_l = 0,\ l = 1, \ldots, m \\ \eta = \eta_{l+1},\ x_l(\eta_{l+1}) = b, \quad \alpha_0 + \alpha_1\eta_{l+1} = \pi \end{gathered} \tag{7.1}$$

It hence follows that

$$x_l(\eta) = \frac{a+b}{2} + \frac{a-b}{2}\cos\eta_{0l}, \quad \eta_{0l} = \frac{\eta - \eta_l}{\eta_{l+1} - \eta_l}\pi \tag{7.2}$$

The argument of the cosine in (7.2) varies between 0 and π on L_l. If the discrete points η_0 are subject to the condition

$$\eta_{0i} = \frac{2i-1}{2n}\pi, \quad i = 1, \ldots, n$$

then these will be roots of the Chebyshev polynomial $T_n(x_0)$ of the first kind $(x_{0i} = \cos\eta_{0i})$, where $S'(\eta_{0i}) \neq 0$ if a fixed design point is at the angular point.

On approximating the curve L_l at $k+1$ points for the parametric assignment of the coordinate $x_l(\eta)$ we obtain in place of (7.1)

$$x_l(\eta) = \sum_{m=0,1}^{k} c_{lm}\cos^m\eta_{0l} \tag{7.3}$$

where all the c_{lm} are determined under the condition that its value of the parameter η corresponds to each $k+1$ point of L_l. For example, a change in η along L_l in proportion to the length of a section of L_l during traversal of Γ from the point $x_l(\eta_l)$ to the point $x_l(\eta)$ satisfies this.

Parametric assignment of the contour Γ by means of (7.1) and (7.3) (the coordinate y is analogous to x) ensures equality of the limits of the first derivatives $x'(\eta)$, $y'(\eta)$ from the left and right at all angular points of the contour Γ since for all L_l the equalities

$$x_l'(\eta_l) = x_l'(\eta_{l+1}) = y_l'(\eta_l) = y_l'(\eta_{l+1}) = 0 \tag{7.4}$$

will be satisfied at the end points according to (7.3).

Therefore, an example is thereby given for smoothing the kernels A, B for $\eta \neq \xi$.

8. The desired characteristics on the contour Γ are determined in the problems under consideration by evaluation of Cauchy-type integrals by means of Kolosov-Muskhelishvili formulas /5, 6/.

Following /9/, we agree not to give a definite value to the Cauchy-type integral over the complex contour $\Gamma(\Gamma = L_1 + \ldots + L_m)$ at singularities of Γ. A Cauchy-type integral at all smooth points, as well as its limits from the left and right at singularities of Γ, can be evaluated by the rectangle formula (6.1). The convergence of (6.1) is discussed in Sect.6. In addition, it should be noted that by approximating ω in the evaluation of a Cauchy-type integral, the number of partition points of Γ can be taken to be substantially greater than n used in solving the SIE. Thus, for instance, $n = 16 - 160$ in the problem considered while the number of points in evaluating Cauchy-type integrals by the rectangle formula could sufficiently be taken an order of magnitude higher. A further increase in the number of points (by two orders) would not change the value of the integral). This is also verified in solving other problems of the mechanics of a continuous medium /14/.

The applicability of the trapezoid and Simpson formulas for the direct evaluation of Cauchy-type singular integrals should be noted.* (*Matveyev A.F., On selfregulation of the problem of evaluating singular integrals with Cauchy and Hilbert kernels in the metric C. Preprint 165, Inst. Theor. Exper. Physics, Moscow, 1982). Consequently, the main method of increasing the accuracy of their evaluation is to reduce the numerical integration step.

Application of a formula obtained earlier /11/

$$\int_L \frac{\omega(\tau)}{\tau - z} d\tau = \sum_{k=0}^{2n} \frac{\omega(\tau_k)}{\tau_k - z} \frac{2\pi i \tau_k}{2n+1} \left[1 - \begin{cases} (z/\tau_k)^{n+1}, & z \in D^+ \\ (\tau_k/z)^n, & z \in D^- \end{cases}\right]$$

where z can approximate the design point on the circle L, is possible if the contour Γ is mapped on L.

9. Calculations were performed for problems in stresses ($k = 2$) with a rectangular domain D. In conformity with the discussion in Sect.3, it is established numerically that the homogeneous system (5.1) has just a trivial solution irrespective of the method of assigning the two allowable matched systems of points on the rectangular contour Γ.

The multilateral tension of a square with side h by a uniform load p applied along its sides, oriented along the coordinate axes, is examined as the first test problem. In this case the right sides of system (5.1) are continuous functions f_{2R} and f_{2I} that vary linearly. The dimensionless values of f_{2R} on the contour Γ, referred to ph, are shown in Fig.1. The graph of the function f_{2I} is obtained by rotating the square with the graphs in Fig.1 counter-clockwise through an angle of $\pi/2$

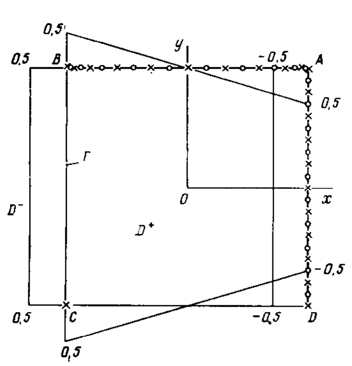

Fig.1

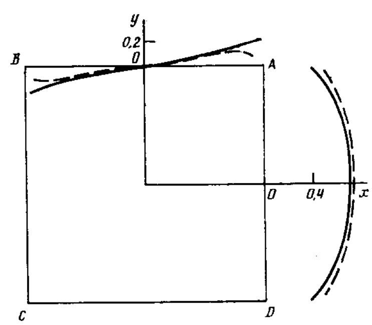

Fig.2

Two different methods of giving the matched systems of points on the contour Γ were considered: for $n = 40$ with a constant step on Γ (shown on the side DA in Fig.1), and by roots of a Chebyshev polynomial of the first kind (shown on the side AB there). In both cases, design points denoted by crosses are placed at the angles of the contour Γ, **discrete** singularities denoted by points are between the PT.

The solution of (5.1) for ω_{Rn}, ω_{In} for $n=40$ is presented in Fig.2 for ω_R (for ω_I the solution is obtained by rotation of ω_R counter-clockwise through $\pi/2$) by dashed lines for the first method of assigning the matched systems of points and by the solid line for the second method.

The function ω_R is symmetric relative to the x axis and antisymmetric with respect to the y axis while the function ω_I possesses the opposite properties. All the assertions of Sect.6 relative to the convergence of the solution and the efficiency of smoothing the SIE **kernels** are confirmed by the graph in Fig.2.

Analysis of the stresses at all points of the square, on its sides, and at the corners by the solution at a system of points given by the second method is ensured with an error not greater than 1% for $n=40$ and is executed according to Sect.8.

It is proved that the solution of a analogous external problem for a rectanglar hole loaded homogeneously along the contour Γ or at infinity reduces to the solution of the SIE (1.4) with right side corresponding to the internal problem (see Fig.1), i.e., the solution ω_{Rn}, ω_{In} agrees with the solution in Fig.2. The stresses on the contour in these cases are determined by two functions $f_{2R}'(t), \operatorname{Re}\varphi'^-(t)$ and have discontinuities of the first kind at angular points.

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